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OPTIMAL STABILIZATION OF ROTATION OF A GYROSTAT IN THE NEWTONIAN FORCE FIELD

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We solve the problem of optimal (in a certain defined sense) stabilization of rotation of a gyrostat (a rigid body with three flywheels) whose center of mass moves along a circular orbit in the central Newtonian force field.

In [1, 2] an analogous problem of stabilization of rotation of a rigid body in inertial motion was solved. Problems of stability of positions of relative equilibrium of stationary motions of rigid bodies and gyrostats in the Newtonian force field were studied in detail in [3–6]. We know that the motions of a rigid body mentioned above can be stabilized by passive damping [7, 8].

1. Initial equations of motion. Statement of the problem. Using the notation of [1] we shall consider a symmetrical gyrostat, i. e. a rigid body with three flywheels ($C_1 = C_2 = C$, $I_1 = I_2 = I$) moving in the central Newtonian force field (O_1 is the center of attraction and O is the center of mass of the gyrostat). Equations of motion of the gyrostat [4, 5] admit the following particular solution of the type of regular precession: the center of mass O moves in the $X_1O_1X_2$ plane along a circular orbit of radius R_0 with constant angular velocity $\Phi^\cdot = \omega_1$. The gyrostat rotates uniformly with relative angular velocity $\varphi^\cdot = \omega$ about the axis of symmetry Ox_3 normal to the orbital plane. Two flywheels whose axes lie in the plane x_1Ox_2 are at rest, and the third flywheel whose axis of rotation is Ox_3 is either at rest or in uniform motion relative to the body. Figure 1 and Table 1 depict the following coordinate systems: $O_1X_1X_2X_3$ is inertial; $Ox_1x_2x_3$ is rigidly connected with the gyrostat and its axes coincide with the axes

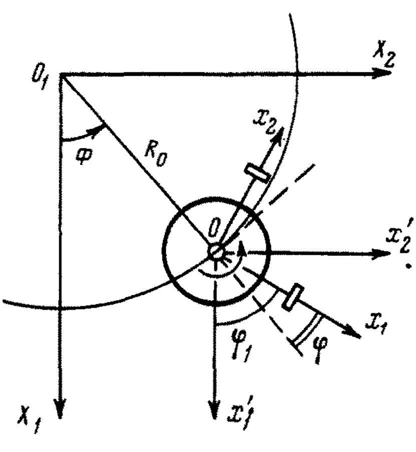

Fig. 1

Table 1

	x_1'	x_2'	x_3'
X_1	β_{11}	β_{12}	β_{13}
X_2	β_{21}	β_{22}	β_{23}
X_3	β_{31}	β_{32}	β_{33}

of the flywheels; $Ox_1'x_2'x_3'$ are the Resal axes (the axis Ox_3' coincides with the axis of symmetry Ox_3 of the gyrostat, Ox_1', Ox_2' lie in the equatorial plane Ox_1x_2 and do not take part in the rotation of the gyrostat about Ox_3. The axes Ox_1', Ox_2' are, in the case of a stationary motion discussed below, parallel to the correponding axes O_1X_1, O_1X_2 of the inertial coordinate system). The dashed lines denote the axes of the orbital coordinate system directed along the radius vector of the center of mass of the gyrostat, tangentially to the orbit and in the direction normal to the orbital plane. Projections of the instantaneous angular velocity of the body p_1, p_2, p_3 on the $Ox_1x_2x_3$ axes and q_1, q_2, q_3 on the $Ox_1'x_2'x_3'$ axes are connected by the following relations:

$$p_1 = q_1 \cos \varphi_1 + q_2 \sin \varphi_1, \quad p_2 = -q_1 \sin \varphi_1 + q_2 \cos \varphi_1, \quad p_3 = q_3 + \varphi_1^{\cdot}$$
$$(\varphi_1^{\cdot} = \varphi^{\cdot} + \Phi^{\cdot} \beta_{33})$$

Let M be the mass of the gyrostat, X_1, X_2, X_3 the coordinates of its center of mass in the $O_1X_1X_2X_3$ system, U the gravitational force function dependent on the coordinates X_1, X_2, X_3 and on the quantities β_{13}, β_{23}, β_{33} characterizing the position of the axis of symmetry Ox_3 of the gyrostat in a stationary space [3]. Using our notation, we have

$$U(X_1, X_2, X_3, \beta_{13}, \beta_{23}, \beta_{33}) = \frac{\varkappa M}{R} + \frac{1}{2} \frac{\varkappa}{R^3} (C_3 - C) - \tag{1.1}$$
$$- \frac{3}{2} \frac{\varkappa}{R^5} (C_3 - C)(X_1\beta_{13} + X_2\beta_{23} + X_3\beta_{33})^2, \qquad R = \sqrt{X_1^2 + X_2^2 + X_3^2}$$

where $\varkappa$ is the gravitational constant.

The equations of motion of the gyrostat [1, 3] have the form

$$MX_i^{\cdot\cdot} = \partial U / \partial X_i \qquad (i = 1, 2, 3) \tag{1.2}$$

$$Cq_1^{\cdot} + (C_3 - C)\, q_2 q_3 + C_3 \varphi_1^{\cdot} q_2 + q_2 g_3 - q_3 g_2 + g_1^{\cdot} = M_{x'_1}$$
$$Cq_2^{\cdot} + (C - C_3)\, q_1 q_3 - C_3 \varphi_1^{\cdot} q_1 + q_3 g_1 - q_1 g_3 + g_2^{\cdot} = M_{x'_2} \tag{1.3}$$
$$C_3 (q_3 + \varphi_1^{\cdot}) + q_1 g_2 - q_2 g_1 + g_3^{\cdot} = M_{x'_3}$$

$$g_1^{\cdot} + Iq_1^{\cdot} + (g_2 + Iq_2)\, \varphi_1^{\cdot} = w_1, \quad g_2^{\cdot} + Iq_2^{\cdot} - (g_1 + Iq_1)\, \varphi_1^{\cdot} = w_2$$
$$g_3^{\cdot} + I_3 (q_3 + \varphi_1^{\cdot})^{\cdot} = w_3 \tag{1.4}$$
$$\beta_{i1}^{\cdot} + q_2 \beta_{i3} - q_3 \beta_{i2} = 0 \quad (i = 1, 2, 3) \quad (1\ 2\ 3) \tag{1.5}$$

Here g_1, g_2, g_3 denote the relative kinetic moments of the flywheels brought to the $Ox'_1x'_2x'_3$ axes, and w_1, w_2, w_3 are the new control moments connected to the old moments u_1, u_2, u_3 by the following relations:

$$w_1 = u_1 \cos \varphi_1 - u_2 \sin \varphi_1, \quad w_2 = u_1 \sin \varphi_1 + u_2 \cos \varphi_1, \quad w_3 = u_3 \tag{1.6}$$

The gravitational force moments $M_{x'_1}$, $M_{x'_2}$, $M_{x'_3}$ have the following form with respect to the $Ox'_1x'_2x'_3$ axes [3]:

$$M_{x_1'} = - \Sigma \frac{\partial U}{\partial \beta_{i3}} \beta_{i2}, \quad M_{x_2'} = \Sigma \frac{\partial U}{\partial \beta_{i3}} \beta_{i1}, \quad M_{x_3'} = 0$$

On the basis of (1.1) we have

$$M_{x_1'} = 3\varkappa (C_3 - C)(\Sigma X_i \beta_{i2})(\Sigma X_i \beta_{i3}) R^{-5}$$
$$M_{x_2'} = -3\varkappa (C_3 - C)(\Sigma X_i \beta_{i1})(\Sigma X_i \beta_{i3}) R^{-5}, \quad M_{x_3'} = 0 \tag{1.7}$$

Let us now replace X_1, X_2, X_3 with spherical coordinates of the center of mass, namely R, Ψ, Φ

$$X_1 = R \cos \Psi \cos \Phi, \quad X_2 = R \cos \Psi \sin \Phi, \quad X_3 = R \sin \Psi$$

We note that the gravitational moments do not appear in Eqs. (1.4) describing the rotation of the flywheels, since the latter are symmetric.

The equations of the stationary motion under investigation are

$$R = R_0,\ R^{\cdot} = 0,\ \Psi = 0,\ \Psi^{\cdot} = 0,\ \Phi = \omega_1 t,\ \Phi^{\cdot} = \omega_1 \tag{1.8}$$
$$\varphi^{\cdot}_1 = \omega_1 + \omega, \quad q_i = 0, \quad \beta_{ik} = \begin{cases} 1 & (i = k) \\ 0 & (i \neq k) \end{cases}$$
$$g_1 = g_2 = 0,\ g_3 = g_3^{\circ} = \text{const},\ w_i = 0 \quad (i, k = 1, 2, 3)$$

Here ω is arbitrary and ω_1 is related to U (1.1) in the following manner [3]:

$$\omega_1 = -\frac{1}{MR_0}\left(\frac{\partial U}{\partial R}\right)_0$$

The subscript zero indicates that the function in question is computed for the stationary mode (1.8). When $\omega = 0$, we obtain the position of relative equilibrium of the gyrostat in a circulat orbit as a particular case.

Equations of motion (1.2)–(1.5) admit (in addition to the trivial ones) three integrals expressing the constancy of the projections of the kinetic moment of the gyrostat on the $O_1X_1X_2X_3$ axes

$$L_i + (Cq_1 + g_1)\beta_{i1} + (Cq_2 + g_2)\beta_{i2} + [C_3(q_3 + \varphi^{\cdot}_1) + g_3]\beta_{i3} = h_i \quad (i = 1, 2, 3) \tag{1.9}$$

where

$$L_1 = MR^2(\Psi^{\cdot}\sin\Phi - \Phi^{\cdot}\sin\Psi\cos\Psi\cos\Phi) \tag{1.10}$$
$$L_2 = -MR^2(\Psi^{\cdot}\cos\Phi + \Phi^{\cdot}\sin\Psi\cos\Psi\sin\Phi),\quad L_3 = MR^2\Phi^{\cdot}\cos^2\Psi$$

are the projections of the kinetic moment vector of the center of mass of the gyrostat on the $O_1X_1X_2X_3$ axes.

Using Eqs. (1.9) we can eliminate [1] g_1, g_2, g_3 from Eqs. (1.4) of rotation of the flywheels. Taking into account (1.2), we obtain

$$(C - I)q_1^{\cdot} = -(C - I)q_2\varphi_1^{\cdot} + (q_3 + \varphi_1^{\cdot})\Sigma(h_i - L_i)\beta_{i2} - q_3\Sigma(h_i - L_i)\beta_{i3} + M_{x_1'} - w_1$$
$$(C - I)q_2^{\cdot} = (C - I)q_1\varphi_1^{\cdot} - (q_3 + \varphi_1^{\cdot})\Sigma(h_i - L_i)\beta_{i1} + q_1\Sigma(h_i - L_i)\beta_{i3} + M_{x_2'} - w_2 \tag{1.11}$$
$$(C_3 - I_3)(q_3 + \varphi_1^{\cdot})^{\cdot} = q_2\Sigma(h_i - L_i)\beta_{i1} - q_1\Sigma(h_i - L_i)\beta_{i2} - w_3$$

Equations (1.2), (1.11) and (1.5) now represent a closed system of transformed equations of motion of the gyrostat in R, Ψ, Φ, q_i, β_{ik} $(i, k = 1, 2, 3)$.

For the stationary mode (1.8) under investigation the constants h_i are given by

$$h_1^{\circ} = h_2^{\circ} = 0, \qquad h^{\circ}_3 = MR_0^2\omega_1 + C_3(\omega_1 + \omega) + g^{\circ}_3$$

Assuming now that the motion (1.8) is unperturbed, let us describe the perturbed motion by

$$R_0 + R,\quad R^{\cdot},\quad \Psi,\quad \Psi^{\cdot},\quad \omega_1 t + \Phi,\quad \omega_1 + \Phi^{\cdot},\quad 1 + \beta_{ik} \quad (i = k)$$
$$\beta_{ik}\ (i \neq k),\quad h_1,\quad h_2,\quad h^{\circ} + h_3,\quad w_i, \quad (i, k = 1, 2, 3)$$
$$(h^{\circ} = h_3^{\circ} - MR_0^2\omega_1)$$

retaining the initial variable notation (h_i are the initial perturbations of the kinetic moment of the gyrostat).

Then on the basis of (1.2), (1.11), (1.5) we obtain the following equations of perturbed motion corresponding to (1.8):

$$Y_i(R, \Psi, \Phi, R^{\cdot}, \Psi^{\cdot}, \Phi^{\cdot}, R^{\cdot\cdot}, \Psi^{\cdot\cdot}, \Phi^{\cdot\cdot}, \beta_{13}, \beta_{23}, \beta_{33}, t) = 0 \quad (i = 1, 2, 3) \tag{1.12}$$
$$q_1^{\cdot} = h_{12}q_3 - (h_{13} + \omega^*)q_2 + \omega^*\Sigma h_{1i}\beta_{i2} +$$
$$+ \beta_{13}\nu\sin 2(\omega_1 t + \Phi) + 2\beta_{23}\nu\sin^2(\omega_1 t + \Phi) + v_1 + Q_1 + U_1$$
$$q_2^{\cdot} = (h_{13} + \omega^*)q_1 - h_{11}q_3 - \omega^*\Sigma h_{1i}\beta_{i1} - \tag{1.13}$$
$$- 2\beta_{13}\nu\cos^2(\omega_1 t + \Phi) - \beta_{23}\nu\sin 2(\omega_1 t + \Phi) + v_2 + Q_2 + U_2$$
$$q_3^{\cdot} = h_{31}q_2 - h_{32}q_1 + v_3 + Q_3$$

$$\beta_{ii}^{\cdot} = B_{ii} \quad (i = 1, 2, 3), \quad \beta_{12}^{\cdot} = -q_3 + B_{12}, \quad \beta_{13}^{\cdot} = q_2 + B_{13} \ (1\,2\,3) \qquad \text{(cont.)}$$

where

$$\frac{h_j}{C - I} = h_{1j}, \quad \frac{h_j}{C_3 - I_3} = h_{3j} \ (j = 1, 2), \quad \frac{h^\circ + h_3}{C - I} = h_{13}, \quad \nu = \frac{3}{2} \frac{\varkappa}{R_0^3} \frac{C_3 - C}{C - I}$$

$$\omega^* = \omega_1 + \omega$$

The control moments v_i are connected with w_i by the following relations:

$$(C - I)\, v_1 = -w_1 + \omega^* h_2, \ (C - I) v_2 = -w_2 - \omega^* h_1, \quad (C_3 - I_3)\, v_3 = -w_3 \tag{1.14}$$

and the terms of at least second order of smallness in

$$R, R^{\cdot}, \Psi, \Psi^{\cdot}, \Phi^{\cdot}, q_i^{\cdot}, \beta_{ik} \ (i, k = 1, 2, 3)$$

have the form

$$B_{i1} = q_3\beta_{i2} - q_2\beta_{i3} \quad (1\,2\,3) \quad (i = 1, 2, 3) \tag{1.15}$$

$$(C - I)\, Q_1 = \Sigma\, [h_i B_{i1} - L_i\, (\omega^* \beta_{i2} + B_{i1})] + U_{1\beta}$$

$$(C - I)\, Q_2 = \Sigma\, [h_i B_{i2} + L_i\, (\omega^* \beta_{i1} - B_{i2})] + U_{2\beta}$$

$$(C_3 - I_3)\, Q_3 = \Sigma\, (h_i - L_i)\, B_{i3}$$

$$L_1 = M\,(R_0 + R)^2\, [\Psi^{\cdot} \sin(\omega_1 t + \Phi) - (\omega_1 + \Phi^{\cdot}) \sin\Psi \cos\Psi \cos(\omega_1 t + \Phi)]$$
$$L_2 = -M\,(R_0 + R)^2\, [\Psi^{\cdot} \cos(\omega_1 t + \Phi) + (\omega_1 + \Phi^{\cdot}) \sin\Psi \cos\Psi \sin(\omega_1 t + \Phi)] \tag{1.16}$$

$$L_3 = M\,(R_0 + R)^2\,(\omega_1 + \Phi^{\cdot}) \cos^2\Psi - M R_0^2 \omega_1$$

$$U_1(R, \Psi, \Phi, t) = \frac{3\varkappa}{2\,(R_0 + R)^3} \frac{C_3 - C}{C - I} \sin 2\Psi \sin(\omega_1 t + \Phi) \tag{1.17}$$

$$U_2(R, \Psi, \Phi, t) = -\frac{3\varkappa}{2\,(R_0 + R)^3} \frac{C_3 - C}{C - I} \sin 2\Psi \cos(\omega_1 t + \Phi)$$

Here L_i denotes the perturbations of the kinetic moment of the center of mass of the gyrostat (1.10); $U_{1\beta}$, $U_{2\beta}$ denote the terms of at least second order of smallness arising from the presence of the gravitational moments (1.7) and vanishing at $\beta_{ik} = 0$ $(i, k = 1, 2, 3)$. The terms U_1, U_2 are also governed by (1.7), but depend only on the perturbations in the orbit of the center of mass of the gyrostat.

Equations of motion (1.12) of the center of mass are not written out in full since their explicit form is not essential in the arguments to follow.

We formulate the problem of optimal stabilization in the following manner: to select the controls v_i as functions of the variables q_i, β_{ik} describing the motion of the gyrostat about the center of mass (and possibly of the variables R, Ψ, Φ) so that the zero solution

$$R = 0, \ \Phi = 0, \ \Psi = 0, \ R^{\cdot} = 0, \ \Psi^{\cdot} = \Phi^{\cdot} = 0$$
$$q_i = 0, \quad \beta_{ik} = 0, \quad h_i = 0 \quad (i, k = 1, 2, 3) \tag{1.18}$$

of Eqs. (1.13) is asymptotically stable in some of the variables q_i, β_{ik} and that some functional

$$\int_0^\infty \Omega\,(q_1, q_2, q_3, \beta_{11}, \beta_{12}, \ldots, \beta_{33}, R, \Psi, \Phi, R^{\cdot}, \Psi^{\cdot}, \Phi^{\cdot}, v_1, v_2, v_3, t)\, dt$$

has a minimum.

2. Solution of the problem of stabilization under the condition that the orbit is imperturbable. First we solve the problem under the assumption that the circular orbit of the center of mass of the gyrostat is unperturbed, i.e.

$$R = 0,\ \Psi = 0,\ \Phi = 0,\ R^{\cdot} = 0,\ \Psi^{\cdot} = 0,\ \Phi^{\cdot} = 0,\ L_i = 0,\ U_1 = U_2 = 0 \qquad (2.1)$$

In our opinion such a statement is not devoid of sense, since the perturbations in the circular orbit do not practically affect the conditions of stability of the stationary motions of the rigid bodies and gyrostats [3–5] obtained under the assumption that the orbit is imperturbable.

The linear part of the perturbed motion equations (1.13) with conditions (2.1) differs from the linear part of the corresponding equations obtained in [1] only in the presence of terms possessing periodic coefficients. The latter however substantially influence the character of the solutions obtained.

We shall first consider an approximate system of equations [1] represented by Eqs. (1.13) and (2.1) with the nonlinear terms Q_1°, Q_2°, Q_3° omitted, and possessing the following zero solution

$$q_i = 0,\ \beta_{ik} = 0 \qquad (i,k = 1, 2, 3) \qquad (2.2)$$

(Q_i° denote the terms Q_i (1.15) with conditions (2.1)).

Continuing to use the notation of [1], we shall define the integrand function Ω_1 of the minimized functional as follows:

$$\Omega_1 = F_1(q_1, q_2, q_3, t) + F_2(\beta_{11}, \beta_{12}, \ldots, \beta_{33}, t) + \Sigma n_i v_i^2 + \\ + \Lambda_1(q_1, q_2, q_3, \beta_{11}, \beta_{12}, \ldots, \beta_{33}, t) \qquad (2.3)$$

$$F_1(q_1, q_2, q_3, t) = \Sigma e_{ik}(t)\, q_i q_k \qquad (2.4)$$

Here F_1, F_2 are positive definite quadratic forms with periodic coefficients. We seek the optimal Liapunov function V° in the form

$$2V^\circ = 2\Phi_0 + \Sigma k_i \Phi_{ii} \qquad (2.5)$$

$$2\Phi_0 = -2\Sigma k_i \beta_{ii} + \Sigma m_i q_i^2 + 2q_1 \Sigma a_{ik}(t)\beta_{ik} + 2q_2 \Sigma b_{ik}(t)\beta_{ik} + 2q_3 \Sigma c_{ik}(t)\beta_{ik}$$

$$\Phi_{kl} = \beta_{kl} + \beta_{lk} + \Sigma \beta_{ki}\beta_{li} = 0 \qquad (k,l = 1, 2, 3;\ k \leqslant l)$$

Using the theorems due to Krasovskii [9] and Rumiantsev [10] we arrive at the following partial differential equation for V°

$$\frac{\partial V^\circ}{\partial t} - \sum \frac{1}{4n_i}\left(\frac{\partial V^\circ}{\partial q_i}\right)^2 + \frac{\partial V^\circ}{\partial q_1}\Big[h_{12}q_3 - (h_{13} + \omega^*)q_2 + \\ + \omega^* \sum h_{1i}\beta_{i2} + \beta_{13}\nu \sin 2\omega_1 t + 2\beta_{23}\nu \sin^2 \omega_1 t\Big] + \\ + \frac{\partial V^\circ}{\partial q_2}\Big[(h_{13} + \omega^*)q_1 - h_{11}q_3 - \omega^* \sum h_{1i}\beta_{i1} - 2\beta_{13}\nu \cos^2 \omega_1 t - \\ - \beta_{23}\nu \sin 2\omega_1 t\Big] + \frac{\partial V^\circ}{\partial q_3}(h_{31}q_2 - h_{32}q_1) + \left(\frac{\partial V^\circ}{\partial \beta_{32}} - \frac{\partial V^\circ}{\partial \beta_{23}}\right) q_1 + \\ + \left(\frac{\partial V^\circ}{\partial \beta_{13}} - \frac{\partial V^\circ}{\partial \beta_{31}}\right) q_2 + \left(\frac{\partial V^\circ}{\partial \beta_{21}} - \frac{\partial V^\circ}{\partial \beta_{12}}\right) q_3 + \sum \frac{\partial V^\circ}{\partial \beta_{ik}} B_{ik} + \\ + F_1(q_1, q_2, q_3, t) + F_2(\beta_{11}, \beta_{12}, \ldots, \beta_{33}, t) + \Lambda_1(q_1, q_2, q_3, \beta_{11}, \beta_{12}, \ldots, \beta_{33}, t) = 0 \qquad (2.6)$$

Collecting the coefficients of like second order terms, we obtain a system of linear differential equations which yield a_{ik}, b_{ik}, c_{ik}, e_{ik} as functions of time and the basic parameters k_i, m_i, n_i $(i, k = 1, 2, 3)$. In the course of solving these equations we ought to choose the simplest particular solutions. Thus for a_{ik}, b_{ik}, c_{ik} (except for those corresponding to the indices 13 and 23) we obtain constant values from the solutions given in [1] by replacing ω with ω^*.

Solutions for a_{j3}, b_{j3}, c_{j3} $(j = 1, 2)$ are the sums of the constants a_{j3}^*, b_{j3}^*, c_{j3}^* and of

$2\omega_1$-periodic functions, the latter controlled by the central force field

$$a_{j3} = a_{j3}^* + K_{j3}\cos 2\omega_1 t + L_{j3}\sin 2\omega_1 t$$
$$b_{j3} = b_{j3}^* + M_{j3}\cos 2\omega_1 t + N_{j3}\sin 2\omega_1 t \qquad (2.7)$$
$$c_{j3} = c_{j3}^* \qquad (j = 1, 2)$$

The relevant calculations are fairly straightforward.

The coefficients $e_{ik}(t)$ of the form F_1 are given by

$$d_1^2 n_1 + a_{23} - a_{32} = e_{11}, \quad d_2^2 n_2 + b_{31} - b_{13} = e_{22}, \quad d_3^2 n_3 + c_{12} - c_{21} = e_{33}$$
$$(h_{13} + \omega^*)(m_1 - m_2) - a_{13} + a_{31} - b_{32} + b_{23} = 2e_{12}$$
$$-h_{12}m_1 + h_{32}m_3 - a_{21} + a_{12} - c_{32} + c_{23} = 2e_{13} \qquad (2.8)$$
$$h_{11}m_2 - h_{31}m_3 - b_{21} + b_{12} - c_{13} + c_{31} = 2e_{23} \quad (d_i = m_i/2n_i;\ i = 1,2,3)$$

The solutions $a_{ik}, b_{ik}, c_{ik}, e_{ik}$ obtained are such that for sufficiently large d_i the functions V° (2.5) and F_1 (2.4) are positive definite.

The form $F_2(\beta_{11}, \beta_{12}, \ldots, \beta_{33}, t)$ is obtained in accordance with (2.5) and (2.6) in the form

$$F_2 = \frac{1}{4n_1}(\Sigma a_{ik}\beta_{ik})^2 + \frac{1}{4n_2}(\Sigma b_{ik}\beta_{ik})^2 + \frac{1}{4n_3}(\Sigma c_{ik}\beta_{ik})^2 + \qquad (2.9)$$
$$+ [\omega^*\Sigma h_{1i}\beta_{i1} + \nu\beta_{13}(1 + \cos 2\omega_1 t) + \nu\beta_{23}\sin 2\omega_1 t]\,(\Sigma b_{ik}\beta_{ik}) -$$
$$- [\omega^*\Sigma h_{1i}\beta_{i2} + \nu\beta_{13}\sin 2\omega_1 t + \nu\beta_{23}(1 - \cos 2\omega_1 t)]\,(\Sigma a_{ik}\beta_{ik})$$

We find its sign by relacing [1] the dependent variables β_{ik} with the independent ones, namely Krylov's angles θ and ψ

$$\beta_{13} = -\beta_{31} = \psi + \ldots, \quad \beta_{32} = -\beta_{23} = \theta + \ldots, \quad \beta_{21} = 0 \qquad (2.10)$$

(where the dots denote terms of higher order of smallness).

Assuming for simplicity that $k_i = k,\ m_i = m,\ n_i = n,\ d_i = d\ (i = 1, 2, 3)$, and that d is sufficiently large, we can write the principal terms of the solutions a_{ik}, b_{ik}, c_{ik} just obtained as

$$a_{13} = \frac{m\nu}{d}\sin 2\omega_1 t + \ldots, \qquad b_{13} = \frac{k - m\nu}{d} - \frac{m\nu}{d}\cos 2\omega_1 t + \ldots \qquad (2.11)$$
$$a_{23} = -\frac{k - m\nu}{d} - \frac{m\nu}{d}\cos 2\omega_1 t + \ldots, \qquad b_{23} = -\frac{m\nu}{d}\sin 2\omega_1 t + \ldots$$
$$b_{31} = -\frac{k + mh_{13}\omega^*}{d} + \ldots, \qquad a_{32} = \frac{k + mh_{13}\omega^*}{d} + \ldots$$

(the remaining coefficients of a_{ik}, b_{ik}, c_{ik} either being equal to zero, or beginning with terms containing in their denominators d of degree higher than the first). With (2.10), (2.11) taken into account the form (2.9) becomes

$$F_2(\theta, \psi, t) = A_1(\psi^2 + \theta^2) + A_2(\psi\sin\omega_1 t + \theta\cos\omega_1 t)^2 + A_3(\psi\cos\omega_1 t -$$
$$- \theta\sin\omega_1 t)^2 + \varepsilon\Phi_2(\theta, \psi, t) \qquad (2.12)$$
$$A_1 = \frac{(2k - m\nu)^2 - (mh_{13}\omega^*)^2}{2dm} - \frac{2k\nu}{d} + 3n\nu^2$$
$$A_2 = \frac{2\nu}{d}(2k - m\nu), \qquad A_3 = \frac{2\nu}{d}(2k + mh_{13}\omega^* - 2m\nu)$$

Here ε is a small parameter, the dots denote the terms of at least third order of smallness which do not affect the sign of F_2, the function Φ_2 is a quadratic form in ψ and θ. The function F_2 is positive definite [9, 11] provided that $A_j > 0\ (j = 1, 2, 3)$, i.e.

$$(2k - m\nu)^2 - (mh_{13}\omega^*)^2 - 4km\nu + 3m^2\nu^2 > 0 \qquad (2.13)$$
$$\nu (2k - m\nu) > 0, \quad \nu (2k + mk_{13}\omega^* - 2m\nu) > 0$$

which yield the following inequalities under the assumption that $k > 0$, $m > 0$

$$\nu > 0, \qquad 2k - m\nu > 0, \qquad 2k + mh_{13}\omega^* - 2m\nu > 0 \qquad (2.14)$$
$$(2k - m\nu)^2 - (mh_{13}\omega^*)^2 - 4km\nu + 3m^2\nu^2 > 0$$

The first inequality expresses the connection between the moments of inertia of the gyrostat

$$C_3 > C \qquad (2.15)$$

(an analogous condition is obtained in [6] for the conventional stability of the body in the case when $\omega = 0$), while the remaining inequalities connect the quantities ω^*, h_3 with the initial parameters k, m. We note that in a real problem where R_0 is large, ν is sufficiently small (of the order of $1/R_0^3$). Assuming for simplicity that $g_3^\circ = 0$ (the flywheels in the unperturbed motion (1.8) are at rest) we find that for $\nu \to 0$ the second inequality of (2.14) is satisfied identically, while the third and fourth inequality lead to the relation [1]

$$| h_{13}\omega^* | < 2k/m \qquad (2.16)$$

which for fixed k and m, restricts the choice of the angular velocities ω^*, and of the region in which the initial perturbations h_3 are admissible.

The inequality (2.15), however, which is influenced by the central force field remains valid, and this is where the real difference lies between the present problem and the problem studied earlier in [1]. Taking into account the remark made above concerning the functions V° and F_1, we conclude that for sufficiently large d and under conditions (2.14), (2.15), the functions (2.3), (2.5) obtained satisfy all the requirements of the theorems in [9, 10], and consequently solve the problem of optimal stabilization of the motion (2.2) by virtue of the approximate system of equations obtained from (1.13), (2.1) by setting $Q_1^\circ = Q_2^\circ = Q_3^\circ = 0$. The principal terms of the function Ω_1 (2.3) should be taken in the form

$$\Lambda_1 = -\Sigma (q_1 a_{ik} + q_2 b_{ik} + q_3 c_{ik}) B_{ik}$$

With (1.6), (1.14) and

$$v_i^\circ = -\frac{1}{2n_i} \frac{\partial V^\circ}{\partial q_i} \qquad (i = 1, 2, 3)$$

taken into account, the required optimal control has the form

$$u_1^\circ = \omega^* (h_2 \cos \omega^* t - h_1 \sin \omega^* t) + (C - I) \Big[d_2 q_2 \sin \omega^* t +$$
$$+ d_1 q_1 \cos \omega^* t + \frac{1}{2n_1} (\Sigma a_{ik} \beta_{ik}) \cos \omega^* t + \frac{1}{2n_2} (\Sigma b_{ik} \beta_{ik}) \sin \omega^* t \Big]$$
$$u_2^\circ = -\omega^* (h_2 \sin \omega^* t + h_1 \cos \omega^* t) + (C - I) \Big[d_2 q_2 \cos \omega^* t -$$
$$- d_1 q_1 \sin \omega^* t - \frac{1}{2n_1} (\Sigma a_{ik} \beta_{ik}) \sin \omega^* t + \frac{1}{2n_2} (\Sigma b_{ik} \beta_{ik}) \cos \omega^* t \Big]$$
$$u_3^\circ = (C_3 - I_3) \left(d_3 q_3 + \frac{1}{2n_3} \Sigma c_{ik} \beta_{ik} \right) \qquad (2.17)$$

It can easily be seen that the optimal Liapunov function (2.5), and hence the optimal control (2.17) with the conditions (2.14) and (2.15), solve the problem of stabilization of the motion (2.2) by virtue of the nonlinear equations (1.13) and (2.1) if the integrand function (2.3) is replaced by

$$\Omega^\circ = \Omega_1 - \sum \frac{\partial V^\circ}{\partial q_i} Q_i^* \qquad (2.18)$$

the latter possessing additional terms of at least third order of smallness.

3. Complete solution of the problem of stabilization. Let us consider the exact equations of perturbed motion (1.13) with respect to the variables R, Ψ, Φ, q_i, β_{ik}, $(i, k = 1, 2, 3)$ and write the required control v_i as a sum of two controls

$$v_i = v_i^\circ + v_i^* \qquad (u_i = u_i^\circ + u_i^*) \qquad (i = 1, 2, 3) \tag{3.1}$$

one of which depends only on the phase coordinates of the stabilized body, and the other only on the perturbations R, Ψ, Φ of the orbit. We shall call the second equation corrective and set it in advance

$$v_j^* = -U_j \qquad (j = 1, 2), \qquad v_3^* = 0 \tag{3.2}$$

We shall seek the basic control v_i° according to the method developed earlier and use the Liapunov function (2.5) obtained in Sect. 2, i.e. we shall apply the corresponding control u_i° as given by (2.17). The integrand function of the minimized functional will become

$$\Omega = \Omega_1 - \sum \frac{\partial V^\circ}{\partial q_i} Q_i \tag{3.3}$$

Here Q_i are determined in accordance with (1.15), (1.16). Function Ω must be positive definite in q_i, β_{ik}. However, it is now also dependent on the variables $R, \Psi, \Phi, R^\cdot, \Psi^\cdot$, $\Phi^\cdot$ which appear in the terms of at least third order of smallness and may, in principle, disturb the sign definiteness of Ω_1 (2.3). The principal part of Ω_1 consists of a sign definite quadratic form in q_i, β_{ik}, while the principal part of the sum appearing in (3.3) contains a quadratic form in q_i, β_{ik} of variable signature, whose coefficients are analytic functions of perturbations R, Ψ, $R^\cdot$, $\Psi^\cdot$, $\Phi^\cdot$ vanishing when $R = 0$, $\Psi = 0$, $R^\cdot = 0$, $\Psi^\cdot = \Phi^\cdot = 0$. When the perturbations are small the above coefficients are arbitrarily small, consequently the function Ω is positive definite [9] in q_i, β_{ik}. So the control (2.17), (3.1), (3.2) is the solution of our problem of stabilization of motion (1.18), provided that the center of mass of the gyrostat moves along a stable circular orbit.

To prove that the motion (1.18) is stable in $R, \Psi, R^\cdot, \Psi^\cdot, \Phi^\cdot$, we shall use the reduction principle [9, 12], regarding the variables $R, \Psi, R^\cdot, \Psi^\cdot, \Phi^\cdot$ as critical, and q_i, β_{ik} as noncritical. According to this principle, the problem of stability can be solved using the "abridged" system of equations in R, Ψ, $R^\cdot$, $\Psi^\cdot$, $\Phi^\cdot$ given by (1.12) with the following conditions :

$$q_i = 0, \qquad \beta_{ik} = 0 \qquad (i, k = 1, 2, 3) \tag{3.4}$$

The abridged system of equations describes therefore the motion of the gyrostat in a central force field in the absence of internal controls dependent on q_i, β_{ik}. Stability of the zero solution

$$R = 0, \ \Psi = 0, \ R^\cdot = 0, \ \Psi^\cdot = \Phi^\cdot = 0 \tag{3.5}$$

of this system with respect to R, Ψ, $R^\cdot$, $\Psi^\cdot$, $\Phi^\cdot$ can easily be established with use of the Liapunov function [3]

$$W = W_1 - \omega_1 W_2 + \frac{c}{MR^2} W_2^2 \tag{3.6}$$

composed of the energy W_1 and the kinetic moment W_2 integrals relative to the O_1X_3 axis. Calculations performed indicate that for sufficiently large $c > 0$ the function W is positive definite in all the variables listed above. Stability of the solution (3.5) implies the stability of the motion (1.18) in all variables by virtue of the complete equations (1.13). Thus, after slight alterations, we can use the Liapunov function (2.5) to obtain the control v°_i (3.1). We note that the motion (1.8), (1.18) considered is not stable in Φ since an arbitrarily small perturbation $\Phi^\cdot = \Delta\omega_1$ leads to the increase in the value of Φ according to the rule

$$\Phi = (\Delta\omega_1)\, t \tag{3.7}$$

The variable Φ appears only as the argument of the bounded functions $\sin(\omega_1 t + \Phi)$ and $\cos(\omega_1 t + \Phi)$. This does not violate our previous deductions based on the stability of the stationary motion under investigation with respect to the variables $R, \Psi, R^\cdot\ \Psi^\cdot, \Phi^\cdot$, but makes necessary the replacement of the argument $\omega_1 t + \Phi$ of the periodic functions in (1.13) by $(\omega_1 + \Delta\omega_1)\, t$. Therefore instead of the coefficients $a_{j3}, b_{j3}, c_{j3}\ (j = 1, 2)$ of the Liapunov function (2.7) we have $a_{j3}(\Delta\omega_1)$, $b_{j3}(\Delta\omega_1)$, $c_{j3}(\Delta\omega_1)$ where the argument $\omega_1 t$ has been replaced by $(\omega_1 + \Delta\omega_1)\, t$. This also applies to the functions (1.17), (2.3). Consequently the control (2.17), (3.1), (3.2), i.e.

$$u_1 = u_1^\circ(\Delta\omega_1) + (C - I)\,[U_1(\Delta\omega_1)\cos\omega^* t + U_2(\Delta\omega_1)\sin\omega^* t]$$

$$u_2 = u_2^\circ(\Delta\omega_1) + (C - I)\,[-U_1(\Delta\omega_1)\sin\omega^* t + U_2(\Delta\omega_1)\cos\omega^* t]$$

$$u_3 = u_3^\circ(\Delta\omega_1)$$

ensures under the conditions (2.14), (2.15) optimal stabilization of the motion (1.8), (1.18); the integrand function of the minimizing functional has the form (3.3). We note that in practice, when R_0 is very large, the corrective control (3.2), (1.17) stipulated by the perturbability of the orbit of the center of mass of the gyrostat can be arbitrarily small and has no practical significance.

The present paper deals with optimal stabilization of only one of the possible stationary motions of a gyrostat and employs internal control moments, however the method developed makes possible, in principle, solution of the problems on stabilization of various positions of relative equilibrium and of stationary motions of a gyrostat in the Newtonian force field.

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ON THE MOTION OF THE HESS GYROSCOPE

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As we know, the solution of the equations of motion of a heavy rigid body about a fixed point in the Hess' case

$$e_1 \sqrt{A(B-C)} + e_3 \sqrt{C(A-B)} = 0, \quad e_2 = 0$$

(A, B, C are the principal moments of inertia and e_1, e_2, e_3 are the coordinates of the center of gravity of the body) is not reducible to quadratures; it is reduced to the solution of the Ricatti differential equation. This complicates investigation of the corresponding motion considerably.

A general qualitative pattern of motion of a body was first given for the Hess' case by N. E. Zhukovskii [1] and followed in more detail by Kovalev [2, 3] who employed the method of moving hodograph (*).

However both these geometrical interpretations are fairly complicated, and give rise to severe difficulties when it comes to determining the motion of a specific rigid body under concrete initial conditions.

In the present paper we study the Hess' case of the motion of a rigid body under the assumption that at the initial instant a high angular velocity ω_0 about some axis, is imparted to the body. We obtain explicit relations connecting the Euler angles with time, and these enable us to analyze in detail the motion of the Hess gyroscope without much difficulty.

1. We construct the equations of motion of a rigid body in the associated rectangular coordinate system $Oxyz$ whose Ox-axis passes through the center of gravity of the body, while the Oy and Oz axes are chosen in such a manner (this is always possible in the Hess case [4]) that the expression for the kinetic energy of the body becomes

$$2T = a_1 x^2 + a(y^2 + z^2) - 2byz$$
$$a_1 = A_{11}(A_{22}A_{33})^{-1}, \quad a = A_{33}^{-1}, \; b = A_{13}(A_{11}A_{33})^{-1}$$

Here x, y, z are the projections of the kinetic moment of the body on the $Oxyz$ axes, and A_{11}, A_{22}, A_{33}, A_{13} are the components of the corresponding inertia tensor for which the relation $A_{13}^2 = A_{11}(A_{22} - A_{33})$ holds.

We also note the following expressions for the projections ω_1, ω_2, ω_3 of the angular velocity:

$$\omega_1 = -by, \quad \omega_2 = ay, \quad \omega_3 = az$$

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*) See also A. M. Kovalev's "Geometrical investigation of certain solutions of the problem of motion of a body with a fixed point", Candidate's dissertation, Donetsk State Univ., 1969.